define a damping ratio, but θ_H does have obvious exponential decay terms. It can be factored into two parts, one with an overall decay $\exp\{-(\zeta\omega_0+\mu\omega/2)t\}$, the other with $\exp\{-(\zeta\omega_0-\mu\omega/2)t\}$.

This modification of the damping will only be of consequence when $\mu\omega/2$ is at least comparable with $\zeta\omega_0$. Usually this will not be the case. It appears that aircraft with reasonably damped high frequency modes will remain stable for realistic amplitudes of the wave. However, ζ may be very small in special circumstances for modern super-performance aircraft flying at high altitudes. For then $\zeta \rightarrow 0$. It should also be noted that damping may be reduced by effects that can occur simultaneously with the wave disturbance studied here. Thus Curtiss,[11] in another study involving aircraft performance governed by equations with time dependent coefficients, has shown that the damping ζ is reduced during deceleration of a craft. Accordingly, very light damping, while unusual, may be encountered in practice.

Pursuing this limit, it is found that the greatest destabilization occurs in the so-called "second unstable region" of the (a,q) space. This is a region of real μ localized about values of a^2 near unity in the notation of Eq. (22) (i.e., a wave frequency ω that is about twice the oscillation frequency ω_0 of the mode). The region is shown in Fig. 2. Only rather small vaues of $2q/a$ are expected, so a good approximation to E is obtained with

$$E \simeq \{4\omega_0^4(1-\zeta^2)^2+\zeta^2\omega^2\omega_0^2\}^{1/2}u/S \simeq 2\omega_0^2 u/S \tag{25}$$

$$|2q/a| \simeq |2u/S| \ll 1 \tag{26}$$

Air speed S will be close to the speed of sound, and the modal period will not exceed a few seconds. Correspondingly, $|2u/S|$ probably cannot exceed 0.1. Examination of Fig. 2 shows that μ will not be much bigger than 0.03. Examination of Eq. (26), for $\omega=2\omega_0$, shows that the craft motion will only be unstable if $\zeta<\mu$, i.e., if $\zeta<0.03$, which is an abnormally small damping ratio.

In conclusion, this mechanism, which is a parametric destabilization of the higher frequency longitudinal mode, is theroretically possible but unlikely in practice. It could of course be induced by loading the craft to have an abnormally long period of pitching but this is hardly a standard procedure.

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Stochastic Simulation Using Covariance Techniques: Modular Program Package for Nonlinear Missile Guidance

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Introduction

SIMULATIONS are often necessary to assess system performance. In this way, the influence of effects that were not taken into account during system design can be examined. Two of the more important of these effects are nonlinearities and stochastic disturbances. For judging system performance generally, mean values and covariances are the most important. Traditionally, nonlinear systems are simulated by applying *Monte Carlo techniques*. Let the nonlinear system be described by

$$\dot{x}(t)=f[x(t),t]+B(t)w(t) \tag{1}$$

driven by a random process $w(t)$ and with random initial conditions. Using shaping filters, we can assume that $w(t)$ is white noise. To obtain statistically meaningful results, a great number of sample responses are generally needed, leading to a high computational burden. The results are, e.g., mean and standard deviation of system states for specific instants of time.

Covariance techniques are methods for directly calculating analytical approximations for the means and the covariances of system states as functions of time. Instead of calculating many sample responses, Eq. (1) is linearized around its mean value and the resulting (nonlinear) system of equations for the means and the covariances is solved numerically only once. The computing time is thus reduced. For the linearization of Eq. (1), two methods can be applied: Taylor series and statistical linearization. In contrast to the CADET-covariance technique,[3] where the whole of Eq. (1) is statistically linearized, we apply a combination of both methods, which is mathematically easier to use and more efficient to compute. Moreover, this combination allows easy design of modular computer programs structured according to physical subsystems.

Covariance Technique for Stochastic Simulation

We linearize the nonlinear stochastic differential equation (1) with the Taylor series expansion around its actual mean value. But we exclude from the Taylor linearization those points in the mathematical model [Eq. (1)] for which the

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assumption of small deviations from the mean is not valid or which are not differentiable. Let $x := \bar{x} + \Delta x$ where $\bar{x} := E\{x\}$ is the mean value of x. We assume that the nonlinear function system $f(x,t)$ in Eq. (1) has the form

$$f(x,t) = \tilde{f}(x,t) + \sum_i \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ g_i[h_i(x)] \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{2}$$

where f and h_i are functions that can be Taylor linearized and the g_i are scalar functions that have to be statistically linearized. $g_i[h_i(x)]$ is statistically linearized[4] by determining the describing functions N_M and N_R by minimizing

$$E\{(g_i[h_i(x)] - [N_M^i \overline{h_i(x)} + N_R^i \Delta h_i(x)])^2\} \tag{3}$$

Using this result in the Taylor linearization of f and h_i we obtain as a first approximation of Eq. (1)

$$\dot{\bar{x}} + \Delta\dot{x} = \tilde{f}(\bar{x},t) + \tilde{A}\Delta x + \sum_i \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ N_M^i h_i(\bar{x}) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + \sum_i \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ N_R^i H_i \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \Delta x + B\bar{w} + B\Delta w \tag{4}$$

with the Jacobians

$$\tilde{A} = \frac{\partial \tilde{f}}{\partial x}; \qquad H_i = \frac{\partial h_i}{\partial x}$$

From this we obtain for the mean

$$\dot{\bar{x}} = \tilde{f}(\bar{x},t) + \sum_i \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ N_M^i h_i(\bar{x}) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + B\bar{w} \tag{5}$$

and the covariance matrix $P := E\{\Delta x \Delta x^T\}$

$$\dot{P} = AP + PA^T + BQB^T \tag{6}$$

where

$$A := \tilde{A} + \sum_i \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ N_R^i H_i \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}; \quad Q\delta(t-\tau) = E\{\Delta w(t) \Delta w(\tau)^T\}.$$

Examining Eq. (3) we see that for the statistical linearization the probability density function of $h_i(x)$ must be known. A Gaussian distribution is a good assumption in most practical cases. This is justified by the low-pass character of most physical systems and by the central limit theorem. Fortunately, the describing function gains N_M^i and N_R^i are often relatively insensitive with respect to deviations from the Gaussian assumption. This is illustrated for a saturation nonlinearity by our numerical example.

Because of the statistical linearization, the mean and the variance equation systems [Eqs. (5) and (6), respectively], are coupled in both directions. Therefore, we have a nonlinear deterministic differential equation system of the order $[n + \frac{1}{2}n(n+1)]$. The computing time of the Monte Carlo technique is proportional to $n \cdot r$ (r is the number of runs without statistical evaluation) and that for our covariance technique is proportional to $[n + \frac{1}{2}n(n+1)]$. On this basis, a saving of computing time can be expected for $n \leq 2r$.

Modular Program Package for Nonlinear Homing Missile Guidance

The totally coupled systems [Eqs. (5) and (6)] can be structured in order to separate problem-dependent subsystems, which are to be represented by appropriate program modules. Equations (5) and (6) show that we have simple analytic expressions for the state equations. For each physical subsystem, the mean value equations and the corresponding submatrices of A, B, and Q in Eq. (6) can be derived separately. This can be done in problem-oriented subroutines provided by the user. The overall equations (5) and (6) can be combined from these data automatically.

We developed a program package[1] for analyzing nonlinear homing missile guidance loops. Figure 1 defines the kinematic variables and Fig. 2 shows the guidance loop block diagram. In the program package each block corresponds to a subroutine to be programmed by the user. Note the special block representing the lateral acceleration saturation, which will be statistically linearized, assuming a Gaussian distribution for the input. The validation of this assumption is examined in the numerical example. Describing functions of other types of nonlinearities and/or input distributions are tabulated in, e.g., Ref. 5.

Example

We present a realistic low-order example in order to illustrate the type of results from the covariance technique

Table 1 Mean value and standard deviation of the miss distance as a function of a_{max} (Monte Carlo simulation results of 200 runs)

a_{max}, [g]	10	20	∞
$\bar{r}_{min}$, [m]	33.0	18.3	11.4
	(33.26)	(18.6)	(12.2)
$\sigma(r_{min})$, [m]	2.66	2.34	2.39
	(2.91)	(2.36)	(2.13)

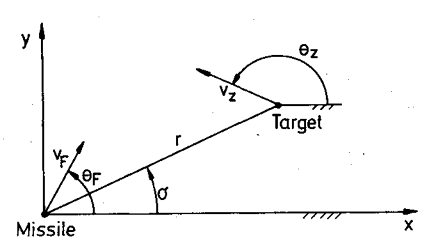

Fig. 1 Definition of kinematic variables for homing missile – target engagement.

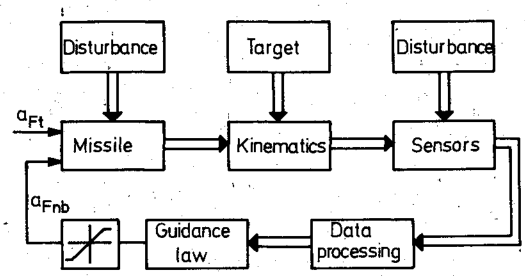

Fig. 2 Block diagram of the guidance loop.

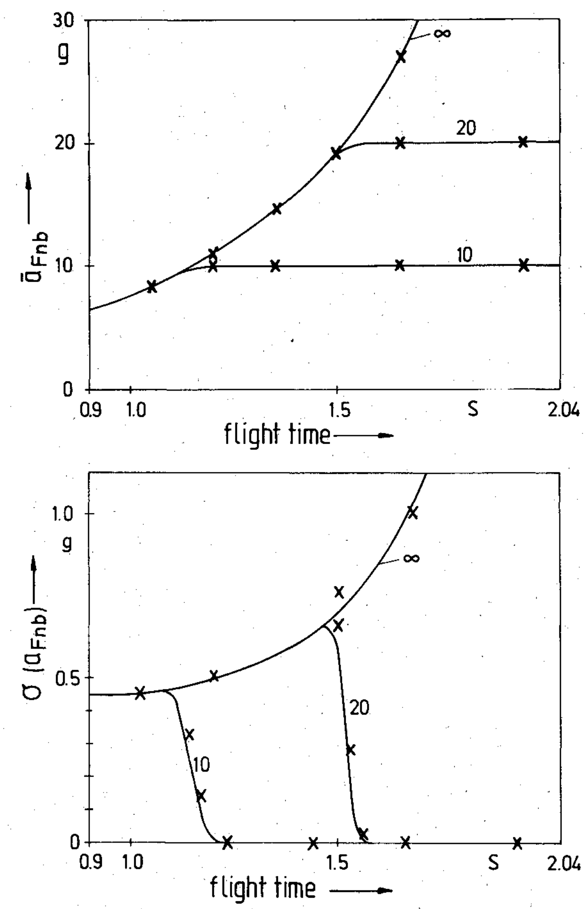

Fig. 3 Time histories of mean value a_{Fnb} and standard deviation $\sigma(a_{Fnb})$ as a function of the parameter a_{max} (crosses represent Monte Carlo simulation results).

and to show the validity of the results compared with those from a Monte Carlo simulation.

We consider a head-on attack with an evasive maneuver of the target. In Fig. 1 we choose for the initial time: $r = 2000$ m, $\sigma = 0$ deg, $v_F = 660$ m/s, $\theta_F = 0$ deg, $v_3 = 330$ m/s, $\theta^z = 180$ deg. The target performs a 5 g lateral acceleration maneuver. The longitudinal acceleration of target and missile are assumed constant. We assume proportional navigation (PN) as the guidance law: commanded lateral acceleration $a_{Fnc} = c \cdot \sigma_g$ with $c = 2000$ rad·m/s. The measured line of sight rate σ_g is assumed to be proportional to the angle error of a radar antenna described by a second-order model

$$[\sigma_g = s(T_1 s + 1)^{-1} (T_2 s + 1)^{-1} (\sigma + \Delta\sigma),$$

$T_1 = 0.1$ s and $T_2 = 0.06$ s].

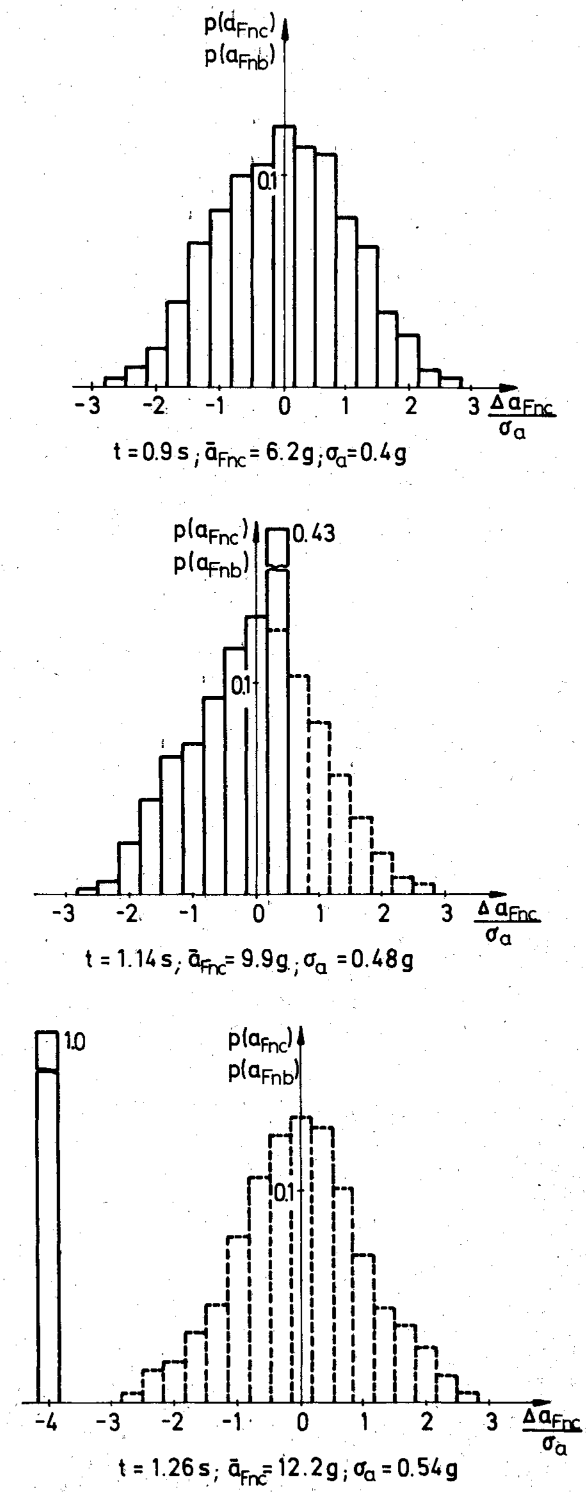

Fig. 4 Probability density functions of a_{Fnc} (dashed lines) and a_{Fnb} for $a_{max} = 10$ g.

The sensor errors $\Delta\sigma = x/r + v$ (glint + thermic noise) are modeled by $\dot{x} = -\alpha x + \alpha w$, v, w white noise, var$(x) = 4$ m^2, $\alpha = 0.1$ s^{-1}, and var$(v) = 2.04 \cdot 10^{-21} \cdot r^4$ rad^2. These are data for a short-range radar.[5] a_{Fnc} is bounded by a_{max}, yielding a_{Fnb}, which is an input to the second-order missile dynamics ($\omega = 10$ rad/s, $\zeta = 0.7$).

Traditionally, in this case the lateral acceleration limiter and nonlinear kinematics made a Monte Carlo simulation necessary. The following results show how our program package, based on a linearized model, reflects the influence of the limiter. First, we show in Table 1 the miss distance r_{min} as a function of the missile lateral acceleration limit a_{max}. The miss distance is the shortest achieved distance between missile and target. Table 1 shows that the covariance technique and the Monte Carlo simulation results for the miss distance agree to within 5%.

With the covariance technique, mean values and standard deviations are computed as functions of time. As an example, Fig. 3 shows these functions for a_{Fnb}, the bounded commanded lateral acceleration, when the limiter becomes effective. These results are validated by 200 Monte Carlo runs, which were statistically analyzed for specific points of time. The accuracy of these results validates the assumptions for the statistical linearization of the limiter (Fig. 4).

Conclusions

Covariance techniques simultaneously using Taylor series and statistical linearization are suggested for stochastic simulation of nonlinear systems as an alternative to Monte Carlo techniques to save computing time. This covariance technique is recommended in cases where the order of the model is not too high. The higher programming effort is overcompensated by the shorter computing times if many production runs are intended.

Using both Taylor and statistical linearization allows easy structuring of the computer programs according to problem-dependent physical subsystems. This makes such simulation programs very flexible.

Our program package was applied for the simulation during feasibility and initial design studies for complex control systems using low-order models. It has also been successfully used within Operations Research (OR)-studies for the simulation of an existing complex system with low-order models, because more exact information was not available.

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